JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 271–299 (1998)
ARTICLE NO. AY986016

Viscous Scalar Conservation Law with Nonlinear Flux Feedback and Global Attractors

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Submitted by Barbara Lee Keyfitz

Received November 25, 1996

In this paper we study the forced viscous scalar conservation law on $(0,1)$ with the nonlinear flux feedback at the boundary. Global existence and uniqueness are established for L^∞ bounded initial conditions and forcing functions. Under an appropriate growth condition on the flux function and nonlinear dissipation at the boundary, we show the existence of an absorbing set that absorbs the whole space $L^\infty(0,1)$, and the existence of a compact global attractor in the L^∞ topology.
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Key Words: Viscous scalar conservation law; nonlinear boundary feedback; L^∞ estimates; absorbing sets; global attractors

[†]Supported in part by Air Force Office of Scientific Research Grant AFOSR F49620-95-1-0437 and AFOSR F49620-95-1-0447.

[‡]Supported in part by the National Science Foundation under grant DMS-9626154.

0022-247X/98 $25.00
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1. INTRODUCTION

We consider in this paper the forced viscous scalar conservation law on $(0,1)$ with nonlinear boundary feedback control:

$$u_t(x,t) - \nu u_{xx}(x,t) + \big(f(u(x,t))\big)_x = F(x,t) \qquad x \in (0,1), \quad t > 0$$
$$\begin{cases} u_x(0,t) - g_0(u(0,t)) = 0 \\ u_x(1,t) + g_1(u(1,t)) = 0 \end{cases} \qquad t > 0 \tag{1.1}$$
$$u(x,0) = u_0(x) \qquad x \in (0,1)$$

where $g_0, g_1 \in C(\mathbb{R})$ are nondecreasing functions with $g_0(0) = g_1(0) = 0$, representing the nonlinear flux feedback controls, $f \in C^1(\mathbb{R})$, $\nu > 0$ is the viscosity constant, $F(x,t)$ is a given body force, and $u_0(x)$ is a given initial state. In particular, (1.1) includes the Burgers equation for $f(u) = \frac{1}{2}u^2$ and zero flux boundary conditions $u_x(0,t) = u_x(1,t) = 0$.

The viscous scalar conservation law on the infinite domain $(-\infty,\infty)$ is well studied. The existence and uniqueness are established in [10, 11] and the references therein, and in [12, 14] the unique entropy solution for the invicid case $(\nu = 0)$ is defined by the vanishing viscosity limit. We also refer to [11] for the general equation of the type (1.1) in a bounded domain in $\mathbb{R}^n$. In [11] the problem is considered in the Hölder spaces, and the existence and uniqueness of classical solutions are shown.

It is the following new and different objectives that motivate this paper. We study the viscous equation for a *general* class of flux boundary conditions on the bounded interval $(0,1)$. The boundary conditions are motivated by the feedback stabilization, i.e., the flux $u_x(\cdot,t)$ is controlled by the *nonlinear* feedback law $-g(u(\cdot,t))$ at the end points, in which the function g determines the stabilization effect.

We assume that the initial condition u_0 and body force F are L^∞ bounded, i.e., $u_0 \in L^\infty(0,1)$ and $F \in L^\infty((0,1)\times(0,\infty))$. It is nontrivial to prove the global existence of solutions and the existence of attractors of solutions in a bounded domain with flux boundary conditions as in (1.1) because of the lack of a "$(f(u)_x, u) \le 0$" type of dissipation property for $u \in H^1(0,1)$.

The contribution of the paper includes the establishment of the L^∞ bounds of solutions for equations in the bounded domain $(0,1)$ with the flux function *f not globally Lipschitz continuous*, and the existence of an exponentially absorbing set for the monotone feedback boundary conditions. In comparison with the results in [11], we have the improved results (in one-dimensional equation as in this paper, which can be extended to the multidimensional case) in the following regards. The initial function u_0

and the forcing term F are in class L^∞ and are not assumed to be smooth. No smoothness of g_i is required, and f is only of class C^1. The case $g_i(u)u \geq 0$ (including $g_i = 0$) (by comparison with $g_i(u)u > 0$) is treated. More importantly, the method we apply is unique. For example, to obtain the absorbing set we use a novel application of Stampacchia-type estimates.

Our approach of showing the global existence of solutions to (1.1) is based on the semigroup approach [3, 18] and the maximum principle. That is, we define a quasi-monotone nonlinear operator A in $L^2(0,1)$ and consider the difference approximation (implicit Euler) of the abstract nonlinear evolution equation $u_t = A(u)$, and then we show that A is maximal by establishing an L^∞ bounded invariant ball of the discrete time flow. Then the convergence of the discrete time solution to the exact solution of (1.1) is established by the nonlinear semigroup theory for the unforced case and Aubin's compactness lemma (e.g., [2]) for the forced case.

We study the effectiveness of the nonlinear feedback mechanism by studying the asymptotic behavior of the solutions as $t \to \infty$. Under the appropriately combined growth conditions on (f, g_1, g_2) (see (4.2)), we establish the absorbing set property [16]; i.e.,

$$-\sigma\left(\sigma_2 e^{-\omega t} + 1\right) \leq u(x,t) \leq \sigma\left(\sigma_1 e^{-\omega t} + 1\right), \tag{1.2}$$

where

$$\sigma_1 = \operatorname*{ess\,sup}_{x\in(0,1)} u_0(x) \qquad \text{and} \qquad \sigma_2 = -\operatorname*{ess\,inf}_{x\in(0,1)} u_0(x),$$

and some $\sigma \geq 1$ and $\omega > 0$ independent of the initial condition. We employ a variant of a Stampacchia-type L^∞ estimate that can be found in [5] to show (1.2). Moreover, we show that there exists $\rho = \rho(\|F\|_{L^\infty})$ with $\rho(0) = 0$ such that

$$\|u(t)\|_{H^1} \leq \rho + \epsilon \qquad \text{for all} \qquad t \geq t_0\left(\|u_0\|_{L^\infty}, \epsilon\right) \tag{1.3}$$

for each $\epsilon > 0$. It thus follows from [16, 6] and (1.2)–(1.3) that the viscous scalar conservation law (1.1) has a nonempty compact maximum attractor $\mathscr{A}$. The case of L^p initial data and forcing functions and the vanishing viscosity limit $\nu \to 0$ will be studied in a forthcoming paper.

The notations used in this paper are standard, and we define $H = L^2(0,1)$, $X = H^1(0,1)$ where X^* is dual space, and $Z = H^2(0,1)$. We denote by $(\cdot,\cdot)$ the inner product in H and $\langle \cdot,\cdot \rangle$ the dual product of $X^* \times X$. Throughout this paper, c is a generic constant.

We refer to control aspect of the Burgers equation to [1, 9, 4] and the references therein. The dynamical system theory we employed in this paper for the infinite dimensional systems is from [16, 17, 6].

2. UNFORCED CASE: EXISTENCE OF CONTRACTION C_0 SEMIGROUP AND STRONG SOLUTIONS

Define a nonlinear map A: $(\mathscr{D}(A) \subset H) \to H$ by

$$A(v)(x) = -\nu v_{xx}(x) + \big(f(v(x))\big)_x \tag{2.1}$$

with the domain

$$\mathscr{D}(A) = \big\{v \in Z = H^2(0,1) | v_x(0) - g_0(v(0)) = 0 \text{ and } v_x(1) + g_1(v(1)) = 0\big\}. \tag{2.2}$$

Integrating by parts, we have, for each $v \in \mathscr{D}(A)$,

$$\begin{aligned}(A(v), w) &= \nu(v_x, w_x) + f(v(1))w(1) - f(v(0))w(0) \\ &\quad + (f(v), w_x) + \nu\big(g_0(v(0))w(0) + g_1(v(1))w(1)\big) \\ &\qquad \forall w \in X = H^1(0,1).\end{aligned} \tag{2.3}$$

In this section, we apply the Crandall−Liggett theorem to obtain the existence of a contraction C_0 semigroup for the unforced (1.1), i.e., with $F \equiv 0$. Because of the lack of global Lipschitz continuity, to obtain monotonicity of the nonlinear map A, we introduce the cut-off function of f for $\gamma > 0$,

$$f_\gamma(y) = \begin{cases} f(\gamma) + f'(\gamma)(y-\gamma) & \text{if } y > \gamma, \\ f(y) & \text{if } |y| \le \gamma, \\ f(-\gamma) + f'(-\gamma)(y+\gamma) & \text{if } y < -\gamma. \end{cases} \tag{2.4}$$

Then $f_\gamma \in C^1(R)$ is globally Lipschitz continuous with the Lipschitz constant,

$$L_\gamma = \sup_{x \in \mathbb{R}} |f'_\gamma(x)| = \sup_{|x| \le \gamma} |f'(x)|. \tag{2.5}$$

We define the corresponding nonlinear map A_γ: $X \to X^*$ by

$$\begin{aligned}\langle A_\gamma(v), w\rangle &= \nu(v_x, w_x) + \big(f_\gamma(v), w_x\big) \\ &\quad + f_\gamma(v(1))w(1) - f_\gamma(v(0))w(0) \\ &\quad + \nu\big(g_0(v(0))w(0) + g_1(v(1))w(1)\big) \qquad \forall w \in X.\end{aligned} \tag{2.6}$$

Then, for each $w \in X$,

$$\left|\left\langle A_\gamma(v), w\right\rangle\right| \le c\Big[\nu\|v_x\|_{L^2} + \|f_\gamma(v)\|_{L^2} + |f_\gamma(v(0))| + |f_\gamma(v(1))| + \nu\big(|g_0(v(0))| + |g_1(v(1))|\big)\Big]\|w\|_{H^1}; \quad (2.7)$$

we know that $A_\gamma(v)$ is a bounded linear functional on X for each $v \in X$, where in (2.7) the continuous Sobolev embedding $H^1(0,1) \hookrightarrow C([0,1])$ is used.

For all $v, w \in X$, since f_γ has the global Lipschitz constant L_γ and g_0 and g_1 are nondecreasing, we have

$$\begin{aligned}
&\left\langle A_\gamma(v) - A_\gamma(w), v - w\right\rangle \\
&\quad = \nu\|v_x - w_x\|_{L^2}^2 + \big(f_\gamma(v) - f_\gamma(w), v_x - w_x\big) \\
&\qquad + \big(f_\gamma(v(1)) - f_\gamma(w(1))\big)(v(1) - w(1)) \\
&\qquad - \big(f_\gamma(v(0)) - f_\gamma(w(0))\big)(v(0) - w(0)) \\
&\qquad + \nu\big(g_0(v(0)) - g_0(w(0))\big)(v(0) - w(0)) \\
&\qquad + \nu\big(g_1(v(1)) - g_1(w(1))\big)(v(1) - w(1)) \\
&\quad \ge \nu\|v_x - w_x\|_{L^2}^2 - L_\gamma\|v - w\|_{L^2}\|v_x - w_x\|_{L^2} - 2L_\gamma\|v - w\|_{L^\infty}^2.
\end{aligned}$$

By the well-known interpolation inequality $\|v\|_{L^\infty} \le c\|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2}$ for all $v \in X$, togethere with Young's inequality, we can find a constant $c(\gamma) > 0$ such that

$$\begin{aligned}
&\left\langle A_\gamma(v) - A_\gamma(w), v - w\right\rangle \\
&\qquad \ge \nu\|v_x - w_x\|_{L^2}^2 - \frac{\nu}{2}\|v - w\|_{H^1}^2 - \frac{c(\gamma)}{\nu}\|v - w\|_{L^2}^2. \qquad (2.8)
\end{aligned}$$

By choosing

$$\omega_\gamma = \omega_\gamma(\nu) = \frac{c(\gamma)}{\nu} + \nu, \qquad (2.9)$$

we obtain, for all $\omega \ge \omega_\gamma$,

$$\left\langle A_\gamma(v) - A_\gamma(w), v - w\right\rangle + \omega\|v - w\|_{L^2} \ge \frac{\nu}{2}\|v - w\|_{H^1}^2 \qquad \forall v, w \in X. \qquad (2.10)$$

Hence we have proved the monotonicity part of the following lemma.

LEMMA 2.1. *Let $g_0, g_1 \in C(\mathbb{R})$ be nondecreasing functions, $f \in C^1(\mathbb{R})$, $\nu, \gamma > 0$ be constants, and A_γ be defined by* (26). *Then for all $\omega \geq \omega_\gamma$, with ω_γ defined by* (2.9), *the nonlinear map $B_{\gamma,\omega} = A_\gamma + \omega I$: $X \to X^*$ is monotone in the sense that $\langle B_{\gamma,\omega}(v) - B_{\gamma,\omega}(w), v - w \rangle \geq 0$ for all $v, w \in X$. Moreover, $B_{\gamma,\omega}$ is hemicontinuous in the sense that $\zeta(\theta) \stackrel{\text{def}}{=} B_{\gamma,\omega}((1-\theta)v + \theta w) \in C([0,1]; X^*)$ for all $v, w \in X$. Finally, $B_{\gamma,\omega}$ is coercive in the sense that $\lim_{\|v\|_{H^1} \to \infty} \langle B_{\gamma,\omega}(v), v\rangle / \|v\|_{H^1} = \infty$.*

Proof. The monotonicity is proved by (2.10). The hemicontinuity is a direct consequence of the definition of $B_{\gamma,\omega}$. The coerciveness is a consequence of (2.10) with $w = 0$. Indeed, $\langle B_{\gamma,\omega}(v), v\rangle/\|v\|_{H^1} \geq (\nu/2) \cdot \|v\|_{H^1} - \|A_\gamma(0)\|_{X^*} \to \infty$ as $\|v\|_{H^1} \to \infty$. The proof is completed.

Next we prove the existence of solutions to the unforced (1.1). To proceed, we restate [15, Theorem 6.5.2 and Corollary].

THEOREM. *Let Y be a reflexive Banach space and T: $Y \to Y^*$ be a coercive hemicontinuous monotone map. Then T is one-to-one and onto.*

By applying the above theorem and the standard elliptic partial differential equation theory (e.g., [8]), we have the following lemma.

LEMMA 2.2. *Assume that the conditions of Lemma* 2.1 *hold true. Then $B_{\gamma,\omega}$: $X \to X^*$ defined in Lemma* 2.1 *is one-to-one and onto, i.e., for each $\xi \in X^*$, there is a unique $\varphi \in X$ such that $B_{\gamma,\omega}(\varphi) = \xi$ in the distributional sense, i.e.,*

$$\left\langle B_{\gamma,\omega}(\varphi), v\right\rangle = \langle \xi, v\rangle \qquad \forall v \in X = H^1(0,1). \tag{2.11}$$

Moreover, if $\xi \in H = L^2(0,1)$, then $\varphi \in Z = H^2(0,1)$, and it satisfies

$$\begin{gathered} \omega\varphi(x) - \nu\varphi_{xx}(x) + \big(f_\gamma(\varphi(x))\big)_x = \xi(x) \\ \varphi_x(0) - g_0(\varphi(0)) = 0, \qquad \varphi_x(1) + g_1(\varphi(1)) = 0, \end{gathered} \tag{2.12}$$

where the first equation holds for a.e. $x \in (0,1)$. *Furthermore, if $\xi \in C([0,1])$, then $\varphi \in C^2([0,1])$ and the first equation holds for every $x \in [0,1]$.*

We need to consider, for $\lambda > 0$ small enough, the restricted nonlinear map $(I + \lambda A)|_{K_\gamma}$: $K_\gamma \to H$, where

$$K_\gamma \stackrel{\text{def}}{=} \{v \in \mathscr{D}(A) \,|\, \|v\|_{L^\infty} \leq \gamma\}. \tag{2.13}$$

LEMMA 2.3. *Assume that the conditions of Lemma* 2.1 *hold true. Assume, in addition, $g_i(0) = 0$ for $i = 0, 1$. Then, for all $\lambda \in (0, 1/\omega_\gamma)$, the range $\mathscr{R}((I + \lambda A)|_{K_\gamma}) \supset \mathrm{Cl}_H(K_\gamma)$, where $\mathrm{Cl}_H(K_\gamma)$ is the closure of K_γ in $H = L^2(0,1)$.*

Proof. We first prove that for each $\xi \in K_\gamma$, there is a $\varphi \in K_\gamma$ such that $(I + \lambda A)(\varphi) = \xi$. Since $\xi \in K_\gamma \subset X^*$, by Lemma 2.2, for each $\lambda \in (0, 1/\omega_\gamma)$, there is a $\varphi \in X$ such that $B_{\gamma, 1/\lambda}(\varphi) = \xi/\lambda$ in the distribution sense (2.11), which yields $(I + \lambda A_\gamma)(\varphi) = \xi$. In particular, because $\xi \in K_\gamma \subset C([0,1])$, Lemma 2.2 implies that there exists a $\varphi \in C^2([0,1])$ satisfying

$$\varphi(x) + \lambda\Big(-\nu\varphi_{xx}(x) + \big(f_\gamma(\varphi(x))\big)_x\Big) = \xi(x) \qquad \forall x \in [0,1]$$
$$\varphi_x(0) - g_0(\varphi(0)) = 0, \qquad \varphi_x(1) + g_1(\varphi(1)) = 0. \tag{2.14}$$

Hence $\varphi \in \mathscr{D}(A)$. To prove $\varphi \in K_\gamma$ and $(I + \lambda A)(\varphi) = \xi$, by the fact that $f_\gamma(\psi) = f(\psi)$ for all $\psi \in K_\gamma$, we need only prove $-\gamma \le \varphi(x) \le \gamma$ for all $x \in [0,1]$. We use the maximum principle.

Indeed, if φ attains its positive maximum at $x_0 \in (0,1)$, since $\varphi_{xx}(x_0) \le 0$ and $(f_\gamma(\varphi(x_0)))_x = f_\gamma'(\varphi(x_0))\varphi_x(x_0) = 0$, (2.14) yields $\max_{x \in [0,1]}\varphi(x) = \varphi(x_0) \le \xi(x_0) \le \gamma$.

Now we suppose that φ attains its positive maximum at $x_0 = 0$. By the boundary conditions in (2.14), $\varphi_x(0) = g_0(\varphi(0)) \ge 0$ because of the fact that $g_0(0) = 0$, $g_0(y)$ is nondecreasing and $\varphi(0) > 0$. But $\varphi_x(0)$ cannot be positive in this case; otherwise it would contradict the assumption that $\varphi(0)$ is a positive maximum. Thus $\varphi_x(0) = 0$ and then $(f_\gamma(\varphi(0)))_x = 0$. Together with (2.14) and the fact that $\varphi_{xx}(0) \le 0$ (otherwise $\varphi(x) \ge \varphi(0)$ for $x > 0$ small enough), we still have $\max_{x \in [0,1]}\varphi(x) = \varphi(0) \le \xi(0) \le \gamma$.

By the same token, if φ attains its positive maximum at $x_0 = 1$, then $\max_{x \in [0,1]}\varphi(x) = \varphi(1) \le \xi(1) \le \gamma$.

Similarly, we can prove that $\varphi(x) \ge -\gamma$ holds for all $x \in [0,1]$. The proof for the case $\xi \in K_\gamma$ is completed.

Now we consider the general case that $\xi \in \mathrm{Cl}_H(K_\gamma)$. We can choose a sequence $\{\xi_n\} \subset K_\gamma$, such that $\xi_n \to \xi$ strongly in $H = L^2(0,1)$. Since $\xi_n \in K_\gamma \subset C([0,1])$ and $\|\xi_n\|_{L^\infty} \le \gamma$, by Lemma 2.2 and the previous arguments, the sequence $\{\varphi_n\} \subset C^2([0,1])$ satisfying

$$\varphi_n(x) + \lambda\Big(-\nu\varphi_{n,xx}(x) + \big(f_\gamma(\varphi_n(x))_x\big)\Big) = \xi_n(x) \qquad \forall x \in [0,1]$$
$$\varphi_{n,x}(0) - g_0(\varphi_n(0)) = 0, \qquad \varphi_{n,x}(1) - g_1(\varphi_n(1)) = 0 \tag{2.15}$$

has the property that $\|\varphi_n\|_{L^\infty} \le \gamma$. By the standard elliptic partial differential equation theory, $\varphi_n \to \varphi$ strongly in $H^1(0,1)$ and $\varphi \in H^2(0,1)$ satisfies (2.12). Hence $\varphi_n \to \varphi$ pointwise and $\|\varphi\|_{L^\infty} \le \sup_n\|\varphi_n\|_{L^\infty} \le \gamma$, which proves $\varphi \in K_\gamma$. Therefore, $f_\gamma(\varphi) = f(\varphi)$. Together with (2.12), we conclude that $(I + \lambda A)(\varphi) = \xi$. The proof is completed.

By applying the Crandall–Liggett Theorem (see, e.g., [17, 13, 3]), we have the following existence of a contraction C_0 semigroup for the unforced (1.1).

THEOREM 2.1. *Assume the conditions of Lemmas* 2.1 *and* 2.3 *hold true. then* $-A$ *generates a nonlinear contraction* C_0 *semigroup* $S(t)$ *on* $L^\infty(0,1) = \bigcup_{m=1}^\infty \mathrm{Cl}_H(K_m)$, *namely,*

$$S(t)(v) = \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-1}(v)$$

$$\text{in } H = L^2(0,1), \quad \forall v \in L^\infty(0,1), \quad \forall t > 0. \quad (2.16)$$

Moreover, $\|S(t)(v)\|_{L^\infty} \le \|v\|_{L^\infty}$ *for all* $v \in L^\infty(0,1)$ *and* $\|S(t)(v) - S(t)(w)\|_{L^2} \le e^{\tilde{\omega}t}\|v - w\|_{L^2}$ *for all* $v, w \in L^\infty(0,1)$ *and* $t \ge 0$, *where* $\tilde{\omega} = \omega_{\max\{\|v\|_{L^\infty}, \|w\|_{L^\infty}\}}$ *is defined by* (2.9).

Proof. By the Crandall–Liggett Theorem, for each $\gamma > 0$, $-A|_{K_\gamma}$ generates a C_0 semigroup $S_\gamma(t)$ on $\mathrm{Cl}_H(K_\gamma)$ such that $\|S_\gamma(t)(v)\|_{L^\infty} \le \gamma$ for all $v \in \mathrm{Cl}_H(K_\gamma)$ and $\|S_\gamma(t)(v) - S_\gamma(t)(w)\|_{L^2} \le e^{\omega_\gamma t}\|v - w\|_{L^2}$ for all $v, w \in K_\gamma$ and $t \ge 0$. Hence $S_\gamma(t)$ is increasing in γ in the sense that $S_{\gamma_2}(t)$ is an extension of $S_{\gamma_1}(t)$ if $\gamma_2 \ge \gamma_1$. Therefore the desired nonlinear contraction C_0 semigroup $S(t)$ exists. The proof is completed.

The definition of the C_0-semigroup (2.16) is in the weak sense, with $u(\cdot, t) = S(t)u_0(\cdot)$ not necessarily differentiable with respect to t. We need to obtain stronger results so that the unforced conservation law has a solution in the classical sense.

By the standard elliptic partial differential equation arguments, A: $(K_m \subset H) \to H$ is a closed nonlinear map. Moreover, $\bigcup_{m=1}^\infty K_m = \mathscr{D}(A)$. Applying [17, Theorem 5.2], we have the following existence of strong solutions of the unforced (1.1).

THEOREM 2.2. *Assume that the conditions of Theorem* 2.1 *hold true. Then the nonlinear contraction semigroup* $S(t)$ *given by Theorem* 2.1 *satisfies* $S(t)(\mathscr{D}(A)) \subset \mathscr{D}(A)$, *and* $S(t)u_0$ *is differentiable in* t *for all* $t > 0$ *provided* $u_0 \in \mathscr{D}(A)$. *In other words, there is a unique strong solution* $u(x,t)$ *of the unforced conservation law*

$$\begin{array}{ll} u_t(x,t) - \nu u_{xx}(x,t) + \big(f(u(x,t))\big)_x = 0 & x \in [0,1], \quad t > 0 \\ \left.\begin{array}{l} u_x(0,t) - g_0(u(0,t)) = 0 \\ u_x(1,t) + g_1(u(1,t)) = 0 \end{array}\right\} & t \ge 0 \\ u(x,0) = u_0(x) & x \in [0,1], \end{array} \quad (2.17)$$

provided $u_0 \in \mathscr{D}(A)$. *Furthermore,* $\|u(\cdot,t)\|_{L^2} \le \|u_0\|_{L^2}$ *for all* $t \ge 0$ *provided* $u_0 \in \mathscr{D}(A)$.

Theorem 2.2 demonstrates that $\|u(\cdot,t)\|_{H^2}$ does not blow up at any finite time t provided $u_0 \in \mathscr{D}(A)$. Thus $\|u(\cdot,t)\|_{H^1}$ and $\|u(\cdot,t)\|_{L^\infty}$ do not blow up at any finite t. But Theorem 2.2 does not give explicit bounds for $\|u(\cdot,t)\|_{H^1}, \|u(\cdot,t)\|_{L^\infty}$, etc., in t. We will study these bounds in Sections 3 and 4. We consider the more general case, the *forced* case.

3. FORCED CASE: GLOBAL EXISTENCE, UNIQUENESS AND REGULARITY

For the forced (1.1), we adopt the following abstract setting:

$$\frac{du}{dt} + A(u) = F, \tag{3.1}$$

with $\mathscr{D}(A)$ and A defined by (2.2) and (2.1). If there is no confusion, we state $u = u(t) = u(\cdot,t) \in \mathscr{D}(A)$ for each $t \geq 0$. Consider the time discretization by using the backward Euler's method,

$$u_{k+1} + hA(u_{k+1}) = u_k + hG_k, \tag{3.2}$$

where $G_k = (1/h)\int_{t_k}^{t_{k+1}} F(\cdot,t)\,dt$ and $t_k = kh$. First we study the existence and some properties of the solutions of (3.2).

LEMMA 3.1. *Suppose the conditions of Theorem* 2.1 *hold true*, $u_0 \in L^\infty(0,1)$ *and* $F \in L^\infty((0,1)\times(0,T))$ *for some finite* $T > 0$. *Let* $\gamma(t) = \|u_0\|_{L^\infty} + t\|F\|_{L^\infty((0,1)\times(0,T))}$. *Then for* $h \in (0, 1/\gamma(T))$, *Eq.* (3.2) *has a unique solution* $u_k \in \mathscr{D}(A)$ *for* $1 \leq k \leq T/h$ *with*

$$\|u_k\|_{L^\infty} \leq \gamma(t_k) \tag{3.3}$$

and

$$\frac{1}{2}\|u_m\|_{L^2}^2 + \frac{1}{2}\sum_{k=1}^{m}\|u_k - u_{k-1}\|_{L^2}^2 + \frac{\nu h}{2}\sum_{k=1}^{m}\|u_{k,x}\|_{L^2}^2 \leq \frac{1}{2}\|u_0\|_{L^2}^2 + t_m M_0(t_m) \tag{3.4}$$

for $1 \leq m \leq T/h$, *where*

$$M_0(t) \overset{\text{def}}{=} \gamma(t)\|F\|_{L^\infty((0,1)\times(0,T))} + \frac{1}{2\nu}\max_{|y|\leq\gamma(t)}|f(y)|^2 + 2\max_{|y|\leq\gamma(t)}|f(y)y|. \tag{3.5}$$

Proof. We prove (3.3) by induction. The case for $k = 0$ is trivial. Suppose that $u_k \in \mathscr{D}(A)$ with $\|u_k\|_{L^\infty} \le \gamma(t_k)$. By Lemma 2.2, for each $\gamma > 0$ and $h \in (0, 1/\omega_\gamma)$ with ω_γ defined by (2.9), we can find a unique solution $u_{k+1} \in \mathscr{D}(A)$ satisfying

$$u_{k+1}(x) + h\Big(-\nu u_{k+1,xx}(x) + \big(f_\gamma(u_k(x))\big)_x\Big) = u_k(x) + hG_k(x) \qquad \text{a.e.} \quad x \in (0,1). \tag{3.6}$$

By the same argument as the proof of Lemma 2.3, we know that

$$\|u_{k+1}\|_{L^\infty} \le \|u_k + hG_k\|_{L^\infty} \le \gamma(t_k) + h\|F\|_{L^\infty((0,1)\times(0,T))} = \gamma(t_{k+1}). \tag{3.7}$$

Choosing $\gamma = \gamma_{k+1}$ in (3.6), we obtain (3.2).

Taking the L^2 inner product of (3.2) with u_{k+1}, integrating by parts, and using Young's inequality and the fact that $yg_j(y) \ge 0$ for all $y \in \mathbb{R}$ and $i = 0, 1$ (since g_i is nondecreasing and $g_i(0) = 0$), we obtain

$$\begin{aligned}\frac{1}{2}\Big(\|u_{k+1}\|_{L^2}^2 - \|u_k\|_{L^2}^2 + \|u_{k+1} - u_k\|_{L^2}^2\Big) + \frac{\nu h}{2}\|u_{k+1,x}\|_{L^2}^2 \\ \le h\left(2 \max_{|y| \le \gamma(t_k)} |f(y)y| + \gamma(t_k)\|F\|_{L^\infty((0,1)\times(0,T))} + \frac{1}{2\nu} \max_{|y| \le \gamma(t_k)} |f(y)|^2\right).\end{aligned} \tag{3.8}$$

Summing up (3.8), (3.4) is proved. The proof is completed.

Next we study (3.1) as the limit of (3.2) as $h \to 0^+$. For this purpose, we define two interpolation functions,

$$\begin{aligned} u_h^{(1)}(t) = u_k \quad \text{and} \quad u_h^{(2)}(t) = u_k + \frac{t - kh}{h}(u_k - u_{k-1}) \\ \text{if } t \in \big((k-1)h, kh\big], \end{aligned} \tag{3.9}$$

where u_k is the solution to the semidiscrete equation (3.2) in the sense of Lemma 3.1. Then one can check that

$$u_h^{(2)}(t) - u_0 = \int_0^t \big(-Au_h^{(1)}(s) + G_h(s)\big)\, ds, \tag{3.10}$$

where $G_h(t) = G_k = (1/h)\int_{t_k}^{t_{k+1}} F(\cdot, s)\, ds$ and

$$\frac{du_h^{(2)}(t)}{dt} = \frac{u_k - u_{k-1}}{h} = -A(u_k) + G_{k-1} \tag{3.11}$$

for $t \in ((k-1)h, kh)$. Then, for any $w \in X = H^1(0,1)$,

$$\begin{aligned}
&\left| \left\langle \frac{du_h^{(2)}(t)}{dt}, w \right\rangle \right| \\
&\qquad = | - \langle A(u_k), w \rangle + \langle G_{k-1}, w \rangle | \\
&\qquad = \Big| - \nu(u_{k,x}, w_x) - (f(u_k), w_x) - f\big(u_k(1)\big)w(1) + f\big(u_k(0)\big)w(0) \\
&\qquad\qquad\qquad - \nu\Big(g_0\big(u_k(0)\big)w(0) + g_1\big(u_k(1)\big)w(1)\Big) + (G_{k-1}, w) \Big| \\
&\qquad \le c\Big(\nu \|u_{k,x}\|_{L^2} + \max_{|y| \le \gamma(T)} |f(y)| \\
&\qquad\qquad + 2\nu \max_{|y| \le \gamma(T)} |g(y)| + \|F\|_{L^\infty((0,1)\times(0,T))} \Big) \|w\|_{H^1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^T \left\| \frac{du_h^{(2)}(t)}{dt} \right\|_{X^*}^2 dt \\
&\qquad \le c\left(\nu h \sum_{k=1}^{[T/h]} \|u_{k,x}\|_{L^2}^2 \right. \\
&\qquad\qquad \left. + T\Big(\max_{|y| \le \gamma(T)} |f(y)| + 2\nu \max_{|y| \le \gamma(T)} |g(y)| + \|F\|_{L^\infty((0,1)\times(0,T))} \Big) \right).
\end{aligned}$$

Combining with Lemma 3.1, we arrive at

$$\begin{aligned}
&u_h^{(2)}(t) \qquad \text{is uniformly bounded in } L^2(0,T;X) \text{ in } h > 0; \\
&\frac{du_h^{(2)}}{dt}(t) \qquad \text{is uniformly bounded in } L^2(0,T;X^*) \text{ in } h > 0.
\end{aligned}$$

By standard arguments, $X \hookrightarrow\hookrightarrow H \hookrightarrow X^*$, where the first embedding is compact. By Aubin's Lemma (see Lemma 8.4 of [2]), there exists a sequence $u_{h_n}^{(2)}(t)$ of $u_h^{(2)}(t)$ and $u \in L^2(0,T;X) \cap H^1(0,T,X^*)$ such that

$$\begin{aligned}
u_{h_n}^{(2)} &\to u && \text{(strongly) in } L^2(0,T;H); \\
u_{h_n}^{(2)} &\rightharpoonup u && \text{(weakly) in } L^2(0,T;X); \\
\frac{du_{h_n}^{(2)}}{dt} &\rightharpoonup \frac{du}{dt} && \text{(weakly) in } L^2(0,T;X^*).
\end{aligned} \tag{3.12}$$

To improve the regularity results in (3.12), we need to strengthen Lemma 3.1 as follows.

LEMMA 3.2. *Suppose that the conditions of Lemma* 2.1 *hold true. Let* $\gamma(t)$ *and* $M_0(t)$ *be as in Lemma* 3.1. *Then for* $h \in (0, 1/\gamma(T))$, *the unique solution* u_k *given by Lemma* 3.1 *satisfies*

$$\|u_{m,x}\|_{L^2}^2 + \sum_{k=m_1}^{m} \|u_{k,x} - u_{k-1,x}\|_{L^2}^2 + \nu h \sum_{k=m_1}^{m} \|u_{k,xx}\|_{L^2}^2$$
$$\leq \|u_{m_1,x}\|_{L^2}^2 + \frac{1}{\nu} \max_{|y| \leq \gamma(t_m)} |f'(y)| \cdot \|u_{m_1}\|_{L^2}^2 + M_1(t_m), \quad (3.13)$$

for $1 \leq m_1 \leq m \leq T/h$, *where*

$$M_1(t) = \frac{2t}{\nu} \|F\|_{L^\infty((0,1)\times(0,T))}$$
$$+ 2t\gamma(t) \max_{|y| \leq \gamma(t)} \left(|g_0(y)| + |g_1(y)|\right) + \frac{2}{\nu} \max_{|y| \leq \gamma(t)} |f'(y)| \cdot M_0(t). \quad (3.14)$$

Proof. Taking the L^2 inner product of (3.2) with $-u_{k+1,xx}$, integrating by parts, and following the similar treatment in the proof of Lemma 3.1, we have

$$\|u_{k+1,x}\|_{L^2} - \|u_{k,x}\|_{L^2}^2 + \|u_{k+1,x} - u_{k,x}\|_{L^2}^2 + \nu h \|u_{k+1,xx}\|_{L^2}^2$$
$$\leq 2\gamma(t_k) \max_{|y| \leq \gamma(t_k)} \left(|g_0(y)| + |g_1(y)|\right)$$
$$+ \frac{2}{\nu}\left(h\|G_k\|_{L^2}^2 + \max_{|y| \leq \gamma(t_k)} |f'(y)|^2 \cdot h\|u_{k+1,x}\|_{L^2}^2 \right).$$

Summing up k and applying (3.4), we obtain (3.13). The proof is completed.

Since $u_k \in \mathscr{D}(A)$ for $1 \leq k \leq T/h$, (3.11) yields

$$\left\| \frac{du_h^{(2)}(t)}{dt} \right\|_{L^2} \leq \| -A(u_k) + G_{k-1} \|_{L^2}$$
$$\leq \nu \|u_{k,xx}\|_{L^2} + \max_{|y| \leq \gamma(T)} |f'(y)| \cdot \|u_{k,x}\|_{L^2} + \|F\|_{L^\infty((0,1)\times(0,T))}.$$

It follows from Lemmas 3.1 and 3.2 that

$$u_h^{(2)}(t) \qquad \text{is uniformly bounded in } L^2_{\mathrm{loc}}(0,T;Z) \text{ in } h > 0;$$
$$\frac{du_h^{(2)}}{dt}(t) \qquad \text{is uniformly bounded in } L^2_{\mathrm{loc}}(0,T;H) \text{ in } h > 0,$$

where $Z = H^2(0,1)$. Combining with the previous arguments, we can find a subsequence of $u_{h_n}^{(2)}(t)$ (still denoted by $u_{h_n}^{(2)}(t)$) such that

$$\begin{aligned} u_{h_n}^{(2)} &\to u && \text{(strongly) in } L^2_{\text{loc}}(0,T;X) \\ u_{h_n}^{(2)} &\rightharpoonup u && \text{(weakly) in } L^2_{\text{loc}}(0,T;Z) \\ \frac{du_{h_n}^{(2)}}{dt} &\rightharpoonup \frac{du}{dt} && \text{(weakly) in } L^2_{\text{loc}}(0,T;H). \end{aligned} \tag{3.15}$$

To pass the limit to the discrete conservation law (3.10), we also need the convergence of $u_{h_n}^{(1)}$, which is given by the following lemma.

LEMMA 3.3. *Let $u^{(1)}$ and $u^{(2)}$ be piecewise constant and piecewise linear functions defined by* (3.9). *Then*

$$\begin{aligned} &\lim_{h\to 0^+} \|u_h^{(2)} - u_h^{(1)}\|_{L^2(0,T;H)} \\ &\qquad = \lim_{h\to 0^+} \|u_h^{(2)} - u_h^{(1)}\|_{L^2(0,T;X)} = 0. \end{aligned} \tag{3.16}$$

Proof. By the definition of $u^{(1)}$ and $u^{(2)}$ in (3.9), we have $\|u_h^{(2)} - u_h^{(1)}\|^2_{L^2(0,T;H)} = (h/3)\sum_{k=1}^{[T/h]} \|u_k - u_{k-1}\|^2_{L^2} + O(h)$, where $[s]$ denotes the largest integer $[s]$ such that $[s] \le s$ and $O(h) \to 0$ as $h \to 0^+$. Thus, by Lemma 3.1, $\lim_{h\to 0^+} \|u_h^{(2)} - u_h^{(1)}\|_{L^2(0,T;H)} = 0$. Again by (3.9), $\|u_h^{(2)} - u_h^{(1)}\|^2_{L^2(\delta,T;X)} = (h/3)\sum_{k=[\delta/h]}^{T/h} \|u_k - u_{k-1}\|^2_{H^1} + O(h)$ for any $\delta \in (0,T)$. But $\int_0^\delta \|u_h^{(1)}(t)\|^2_{L^2}\,dt = h\sum_{k=0}^{[\delta/h]-1} \|u_k\|^2_{L^2} + O(h)$ and $\int_0^\delta \|u_h^{(2)}(t)\|^2_{L^2}\,dt = (4h/3)\sum_{k=0}^{[\delta/h]-1} \|u_k\|^2_{L^2} + O(h)$. Then by Lemmas 3.1 and 3.2, $\lim_{h\to 0^+} \|u_h^{(2)} - u_h^{(1)}\|_{L^2(0,T;X)} = 0$. The proof is completed.

With the preparations in this section, we are ready to prove the existence of solutions for the scalar conservation law (1.1).

THEOREM 3.1. *Suppose the conditions of Theorem* 2.1 *hold true, $F \in L^\infty((0,1)\times(0,T))$ and $u_0 \in X = L^\infty(0,1)$. Then the scalar conservation law* (1.1) *has a unique solution $u(t) \in L^2_{\text{loc}}(0,T;Z) \cap H^1_{\text{loc}}(0,T;H) \cap L^2(0,T;X) \cap H^1(0,T;X^*) \cap L^\infty((0,1)\times(0,T))$ such that*

$$\|u(t)\|^2_{L^2} + 2\nu\int_{t_0}^t \|u_x(s)\|^2_{L^2}\,ds \le \|u(t_0)\|^2_{L^2} + 2tM_0(t), \tag{3.17}$$

for all $t \geq t_0 \geq 0$ *and*

$$\|u_x(t)\|_{L^2}^2 + \nu\int_{t_0}^{t} \|u_{xx}(s)\|_{L^2}^2\, ds \leq \|u_x(t_0)\|_{L^2}^2 + \frac{1}{\nu}\max_{|y|\leq\gamma(t)} |f'(y)|\cdot\|u(t_0)\|_{L^2}^2 + M_0(t) + M_1(t), \tag{3.18}$$

for all $t \geq t_0 > 0$, *where* $M_0(t)$ *and* $M_1(t)$ *are as in Lemmas* 3.1 *and* 3.2.

Proof. By (3.10), we have, for each $v \in C^\infty([0,1]\times[0,T])$,

$$\begin{aligned}\int_0^T \left\langle \frac{du_{h_n}^{(2)}(\cdot,s)}{dt}, v(\cdot,s)\right\rangle ds &= -\nu\int_0^T\Big(g_0\big(u_{h_n}^{(1)}(0,s)\big)v(0,s) + g_1\big(u_{h_n}^{(1)}(1,s)\big)v(1,s)\Big)\,ds \\ &\quad - \nu\int_0^T\Big(u_{h_n,x}^{(1)}(\cdot,s), v_x(\cdot,s)\Big)\,ds + \int_0^T\Big(G_{h_n}(\cdot,s), v(\cdot,s)\Big)\,ds \\ &\quad - \int_0^T\Big(f'\big(u_{h_n}^{(1)}(\cdot,s)\big)u_{h_n,x}^{(1)}(\cdot,s), v(\cdot,s)\Big)\,ds\end{aligned} \tag{3.19}$$

By Lemmas 3.1 and 3.2 and the definition of $u_{h_n}^{(1)}$, we have $\|u_{h_n}^{(1)}(t)\|_{L^\infty} \leq \gamma(t)$ for all $t \in [0,T]$. Then, for $i = 0,1$ and for any $\varepsilon > 0$,

$$\begin{aligned}\left|\int_0^T\Big(g_i\big(u_{h_n}^{(1)}(i,s)\big) - g_i(u(i,s))\Big)\,ds\right| &\leq 2\varepsilon\max_{|y|\leq\gamma(T)}|g_i(y)| \\ &\quad + \max_{|y|\leq\gamma(T)}|g_i'(y)|\sqrt{T}\left(\int_\varepsilon^T\left|u_{h_n}^{(1)}(i,s) - u(i,s)\right|^2 ds\right)^{1/2}.\end{aligned} \tag{3.20}$$

But $u_{h_n}^{(1)} \to u(t)$ strongly in $L^2(\varepsilon,T;X)$ together with the continuous Sobolev embedding $H^1(0,1) \hookrightarrow C([0,1])$, we have, for $i = 0,1$,

$$\begin{aligned}\int_\varepsilon^T\left|u_{h_n}^{(1)}(i,s) - u(i,s)\right|^2 ds &\leq \int_\varepsilon^T\left\|u_{h_n}^{(1)}(\cdot,s) - u(\cdot,s)\right\|_{L^\infty}^2 ds \\ &\leq c\|u_{h_n}^{(1)} - u\|_{L^2(\varepsilon,T;X)}^2 \to 0 \qquad \text{as } n\to\infty.\end{aligned}$$

Thus (3.20) yields

$$\limsup_{n\to\infty}\left|\int_0^T\Big(g_i\big(u_{h_n}^{(1)}(i,s)\big)-g_i(u(i,s))\Big)\,ds\right|\le 2\varepsilon\max_{|y|\le\gamma(T)}|g(y)| \quad (3.21)$$

for arbitrary $\varepsilon>0$. Let $\varepsilon\to 0^+$; we obtain

$$\int_0^T\Big(g_0\big(u_{h_n}^{(1)}(0,s)\big)v(0,s)+g_1\big(u_{h_n}^{(1)}(1,s)\big)v(1,s)\Big)\,ds$$
$$\to\int_0^T\big(g_0(u(0,s))v(0,s)+g_1(u(1,s))v(1,s)\big)\,ds \qquad \text{as } n\to\infty. \quad (3.22)$$

Now by (3.12) and (3.16) and using the fact that $u\in L^\infty((0,1)\times(0,T))$, we can pass to the limit of (3.19) to obtain

$$\begin{aligned}\int_0^T\left\langle\frac{du(\cdot,s)}{dt},v(\cdot,s)\right\rangle ds &= -\nu\int_0^T\big(g_0(u(0,s))v(0,s)+g_1(u(1,s))v(1,s)\big)\,ds\\ &\quad-\nu\int_0^T(u_x(\cdot,s),v_x(\cdot,s))\,ds \qquad (3.23)\\ &\quad-\int_0^T(f'(u(\cdot,s))u(\cdot,s),v(\cdot,s))\,ds\\ &\quad+\int_0^T(F(\cdot,s),v(\cdot,s))\,ds\end{aligned}$$

for all $v\in C^\infty([0,1]\times[0,T])$.

Let us fix $\varepsilon\in(0,T)$. By (3.15), $u\in L^2(\varepsilon,T;Z)\cap L^2(0,T;H)$. Integrating by parts in (3.23), we obtain, for all $v\in C^\infty([0,1]\times[\varepsilon,T])$,

$$\begin{aligned}\int_\varepsilon^T\int_0^1\bigg(\frac{du(x,t)}{dt}&-\nu u_{xx}(x,t)\\ &+f'(u(x,t))u(x,t)-F(x,t)\bigg)v(x,t)\,dx\,dt\\ &=-\nu\int_\varepsilon^T\big[(g_0(u(0,t))-u_x(0,t))v(0,t)\\ &\qquad+(g_1(u(1,t))+u_x(1,t))v(1,t)\big]\,dt. \qquad (3.24)\end{aligned}$$

By taking all possible $v \in C_0^\infty([0,1] \times [\varepsilon, T])$, we know that for a.e. $(x, T) \in (0,1) \times (\varepsilon, T)$,

$$u_t(x,t) - \nu u_{xx}(x,t) + \big(f(u(x,t))\big)_x = F(x,t). \tag{3.25}$$

By straightforward real analysis arguments, for a.e. $t \in (\varepsilon, T)$, (3.25) is true for a.e. $x \in (0,1)$. Hence (3.24) yields, for all $v \in C^\infty([0,1] \times [\varepsilon, T])$,

$$\int_\varepsilon^T \big[(g_0(u(0,t)) - u_x(0,t)) v(0,t) + \big(g_1(u(1,t)) + u_x(1,t)\big) v(1,t) \big]\, dt = 0.$$

Thus, for a.e. $t \in (\varepsilon, T)$,

$$u_x(0,t) - g_0\big(u(0,t)\big) = 0 \qquad u_x(1,t) + g_1\big(u(1,t)\big) = 0. \tag{3.26}$$

But $\varepsilon \in (0, T)$ is arbitrary, and (3.25) and (3.26) hold for a.e. $t \in (0, T)$.

Furthermore, by (3.19) and (3.23), for $v \in C^\infty([0,1] \times [0,T])$ with $v(x,T) = 0$, we have

$$\begin{aligned}
-\Big(u_{h_n}^{(2)}(\cdot,0), v(\cdot,0)\Big) &- \int_0^T \Big(u_{h_n}^{(2)}(\cdot,s), v_t(\cdot,s)\Big)\, ds \\
&= -\nu \int_0^T \Big(g_0\Big(u_{h_n}^{(1)}(0,s)\Big) v(0,s) + g_1\Big(u_{h_n}^{(1)}(1,s)\Big) v(1,s)\Big)\, ds \\
&\quad - \nu \int_0^T \Big(u_{h_n,x}^{(1)}(\cdot,s), v_x(\cdot,s)\Big)\, ds + \int_0^T \Big(G_{h_n}(\cdot,s), v(\cdot,s)\Big)\, ds \\
&\quad - \int_0^T \Big(f'\Big(u_{h_n}^{(1)}(\cdot,s)\Big) u_{h_n,x}^{(1)}(\cdot,s), v(\cdot,s)\Big)\, ds
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
-(u(\cdot,0), v(\cdot,0)) &- \int_0^T (u(\cdot,s), v_t(\cdot,s))\, ds \\
&= -\nu \int_0^T (u_x(\cdot,s), v_x(\cdot,s))\, ds - \int_0^T (f'(u(\cdot,s)) u(\cdot,s), v(\cdot,s))\, ds \\
&\quad - \nu \int_0^T (g_0(u(0,s)) v(0,s) + g_1(u(1,s)) v(1,s))\, ds \\
&\quad + \int_0^T (F(\cdot,s), v(\cdot,s))\, ds.
\end{aligned} \tag{3.28}$$

Note that $u_{h_n}^{(2)}(\cdot, 0) = u_0$ for each n. By passing to the limits of (3.27), and combining with (3.28), we obtain $(u_0, v(\cdot, 0)) = (u(\cdot, 0), v(\cdot, 0))$. But v is arbitrary, and we obtain $u(x, 0) = u_0(x)$ a.e. $x \in (0, 1)$. In addition, since $u \in L^2(0, T; X)$ and $du/dt \in L^2(0, T; X^*)$ and the Sobolev embedding $X \hookrightarrow\hookrightarrow L^\infty(0, 1) \hookrightarrow X^*$ holds, by Lemma 5.5.1 of [15], we have $u \in C(0, T; H)$, and thus

$$u(\cdot, t) \to u(\cdot, 0) = u_0 \qquad \text{in } L^\infty(0, 1) \text{ as } t \to 0^+. \tag{3.29}$$

As a summary, we arrive at

$$\begin{aligned} u_t(x, t) - \nu u_{xx}(x, t) + \left(f(u(x, t))\right)_x = F(x, t) \qquad &\text{a.e. } x \in (0, 1) \\ \left.\begin{aligned} u_x(0, t) - g_0(u(0, t)) = 0 \\ u_x(1, t) + g_1(u(1, t)) = 0 \end{aligned}\right\} & \end{aligned} \tag{3.30}$$

for a.e. $t > 0$ and $u(\cdot, t) \to u_0$ in $L^\infty(0, 1)$ as $t \to 0^+$.

To prove the uniqueness, we assume that $u_1(t)$ and $u_2(t)$ are two solutions of (1.1). Then, by (2.10),

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_2(t) - u_1(t)\|_{L^2}^2 &= -\left\langle A(u_2(t)) - A(u_1(t)), u_2(t) - u_1(t) \right\rangle \\ &\leq \omega \|u_2(t) - u_1(t)\|_{L^2}^2, \end{aligned}$$

where $\omega \geq \omega_\gamma$ as in (2.9), with $\gamma = \max\{\|u_1\|_{L^\infty((0,1)\times(0,T))}, \|u_2\|_{L^\infty((0,1)\times(0,T))}\} < \infty$. By Gronwall's inequality and the fact that $u_1(0) = u_2(0) = u_0$, we obtain $\|u_2(t) - u_1(t)\|_{L^2}^2 \leq e^{\omega T}\|u_2(0) - u_1(0)\|_{L^2}^2 = 0$. Hence $u_2(x, t) = u_1(x, t)$ a.e. $(x, t) \in (0, 1) \times (0, T)$, which completes the proof of uniqueness.

Finally, (3.17) and (3.18) follow from the estimates (3.4) and (3.13) in Lemmas 3.1 and 3.2. The proof of Theorem 3.1 is therefore completed.

At the end of this section, we present some interesting special cases.

EXAMPLE 3.1. Consider the case $f(y) = y^{k+1}/(k+1)$ for an integer $k \geq 0$ and $g_0(y) = g_1(y) = 0$. Then, by Theorem 3.1, we have the global existence, uniqueness, and regularity of the solutions of the following system:

$$\begin{gathered} u_t(x, t) - \nu u_{xx}(x, t) + u^k(x, t) u_x(x, t) = F(x, t) \\ u_x(0, t) = u_x(1, t) = 0 \\ u(\cdot, t) = u_0(x), \end{gathered}$$

provided $u_0 \in L^\infty(0,1)$. A particular case is the Burgers equation ($k = 1$) with homogeneous Neumann boundary conditions. Note that the initial data can be arbitrarily large, as long as it is in $L^\infty(0,1)$.

EXAMPLE 3.2. Let f be as in Example 3.1. Consider the *nonlinear* flux feedback model $g_0(y) = \beta_0 |y|^r y$ and $g_1(y) = \beta_1 |y|^r y$, where $\beta_0, \beta_1 \geq 0$ and $r \geq 0$. Then, by Theorem 3.1, we have the existence, uniqueness, and regularity of the solutions of the following system:

$$u_t(x,t) - \nu u_{xx}(x,t) + u^k(x,t) u_x(x,t) = F(x,t)$$
$$u_x(0,t) - \beta_0 |u(0,t)|^r u(0,t) = 0, \qquad u_x(1,t) + \beta_1 |u(1,t)|^r u(1,t) = 0$$
$$u(x,0) = u_0(x),$$

provided $u_0 \in L^\infty(0,1)$ and $F \in L^\infty((0,1) \times (0,\infty))$. Note again that $k = 1$ gives the Burgers equation, and the initial data can be large. The special case of $r = 0$ is the linear boundary feedback control considered in [1] for the Burgers equation.

We will see, in the following section, that the motivation of considering the *nonlinear* boundary feedback control is that the system is *dissipative*.

4. DYNAMICS: L^∞ ESTIMATES AND GLOBAL L^∞ ATTRACTORS

In this section we study absorbing properties, compactness of the solutions to (1.1), and the existence of global attractors by applying a Stampacchia-type L^∞ estimate. We assume that the conditions of Theorem 3.1 hold true.

We first set

$$\sigma_1 = \operatorname*{ess\,sup}_{x \in (0,1)} \{u_0(x)\} \qquad \text{and} \qquad \lambda(t) = \sigma_1 e^{-\omega t} + 1,$$

where $\omega > 0$ is to be determined below. For $\sigma \geq 1$,

$$u_\sigma(x,t) = \max\{0, u(x,t) - \sigma\lambda(t)\}.$$

Clearly, $u_\sigma(\cdot,t) \in X = H^1(0,1)$ for a.e. $t > 0$ and $u_\sigma(x,0) = 0$ for a.e. $x \in (0,1)$.

Let $p > 2$. Multiplying (1.1) by u_σ^{p-1} and integrating with respect to x over $(0,1)$, we have

$$\frac{1}{p}\frac{d}{dt}\int_0^1 u_\sigma^p\,dx - \omega\sigma\sigma_1 e^{-\omega t}\int_0^1 u_\sigma^{p-1}\,dx + \int_0^1 f'(u)u_x u_\sigma^{p-1}\,dx$$
$$+ \nu\big(g_0(u(0,t))u_\sigma^{p-1}(0,t) + g_1(u(1,t))u_\sigma^{p-1}(1,t)\big)$$
$$+ \nu(p-1)\int_0^1 u_\sigma^{p-2}(u_\sigma)_x u_x\,dx = \int_0^1 F u_\sigma^{p-1}\,dx.$$

We write

$$H(y,\sigma) = \int_0^y f'(s+\sigma)s^{p-1}\,ds \qquad \text{for } y, \sigma \geq 0,$$

then

$$\int_0^1 f'(u)u_x u_\sigma^{p-1}\,dx = H\big(u_\sigma(1,t),\sigma\lambda(t)\big) - H\big(u_\sigma(0,t),\sigma\lambda(t)\big).$$

Also,

$$\int_0^1 u_\sigma^{p-2}(u_\sigma)_x u_x\,dx = \int_0^1 u_\sigma^{p-2}(u_\sigma)_x(u_\sigma)_x\,dx = \frac{4}{p^2}\int_0^1\big((u_\sigma^{p/2})_x\big)^2\,dx.$$

Hence we obtain

$$\frac{1}{p}\frac{d}{dt}\int_0^1 u_\sigma^p\,dx + \frac{4\nu(p-1)}{p^2}\int_0^1\big((u_\sigma^{p/2})_x\big)^2\,dx$$
$$+ \nu\big(g_0(u_\sigma(0,t)+\sigma\lambda(t))u_\sigma^{p-1}(0,t)$$
$$+ g_1(u_\sigma(1,t)+\sigma\lambda(t))u_\sigma^{p-1}(1,t)\big)$$
$$+ H\big(u_\sigma(1,t),\sigma\lambda(t)\big) - H\big(u_\sigma(0,t),\sigma\lambda(t)\big)$$
$$= \int_0^1\big(\omega\sigma\sigma_1 e^{-\omega t} + F\big)u_\sigma^{p-1}\,dx. \tag{4.1}$$

We assume the following growth condition: there exist $\sigma_0 \geq 1$ and $p > 2$ such that for $\sigma \geq \sigma_0$ and $i = 0, 1$,

$$\nu g_i(y + \operatorname{sign}(y)\sigma)|y|^{p-2}y - (-1)^i\int_0^y f'(s + \operatorname{sign}(y)\sigma)|s|^{p-2}s\,ds$$
$$\geq \frac{4\nu(p-1)}{p^2}|y|^p, \tag{4.2}$$

where

$$\operatorname{sign}(y) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

We write

$$\Omega_\sigma(t) = \{x \in (0,1) \,|\, u(x,t) > \sigma\lambda(t)\}$$

for $\sigma \geq \sigma_0$ and define

$$\psi(\sigma) = \sup_{0 \leq s \leq \infty} |\Omega_\sigma(s)|,$$

where $|\Omega_\sigma(s)| = \operatorname{meas}(\Omega_\sigma(s))$.

Define $\|\cdot\|_1$, the equivalent norm of $\|\cdot\|_{H^1}$, by

$$\|v\|_1^2 = v^2(0) + v^2(1) + \int_0^1 v_x^2(x)\,dx \qquad \forall v \in X = H^1(0,1).$$

Then there exist constants $\beta_1, \beta_2 > 0$ such that

$$\|v\|_{L^\infty} \leq \beta_1 \|v\|_1 \quad \text{and} \quad \|v\|_1^2 \geq \beta_2 \|v\|_{L^2}^2 \qquad \forall v \in X.$$

Set $\varphi_\sigma = u_\sigma^{p/2}$. Then, for $\delta > 0$,

$$\begin{aligned} \int_0^1 u_\sigma^{p-1}\,dx &= \int_0^1 |\varphi_\sigma|^{2-2/p}\,dx \leq \|\varphi_\sigma\|_{L^\infty}^{2-p/2} \Omega_\sigma(t) \\ &\leq \frac{\delta^q}{q} \beta_1^2 \|\varphi_\sigma\|_1^2 + \frac{1}{p\delta^p} \Omega_\sigma^p(t), \end{aligned}$$

where we have used Young's inequality with $q = p/(p-1)$. By setting $\delta = (\nu/(p\beta_1))^{(1-p)/p}$, we have

$$\begin{aligned} &\int_0^1 \left(\omega\sigma\sigma_1 e^{-\omega t} + F\right) u_\sigma^{p-1}\,dx \\ &\quad \leq \frac{2\nu(p-1)}{p^2} \|\varphi_\sigma\|_1^2 + \frac{M_1}{p} \left(\left(\sigma_1 \omega e^{-\omega t}\right)^p \sigma^p + \|F\|_{L^\infty}^p\right) \Omega_\sigma^p(t), \end{aligned}$$

where

$$M_1 = \left(\frac{q\nu(p-1)}{p^2\beta_1^2}\right)^{1-p} = \left(\frac{\nu}{p\beta_1}\right)^{1-p}.$$

It thus follows from (4.1) and (4.2) that

$$\frac{d}{dt}\int_0^1 u_\sigma^p\,dx + \omega_1\int_0^1 u_\sigma^p\,dx \le M_1\Big(\big(\sigma_0\omega e^{-\omega t}\big)^p\sigma^p + \|F\|_{L^\infty}^p\Big)\Omega_\sigma^p(t), \quad (4.3)$$

where

$$\omega_1 = \frac{2\beta_2\nu(p-1)}{p}.$$

Multiplying (4.3) by $e^{\omega_1 t}$ and then integrating with respect to t, we obtain

$$\int_0^1 u_\sigma^p\,dx \le \int_0^t e^{-\omega_1(t-s)}M_1\Big(\big(\sigma_1\omega e^{-\omega s}\big)^p\sigma^p + \|F\|_{L^\infty}^p\Big)\Omega_\sigma^p(s)\,ds$$
$$\le \frac{M}{\omega_1}\Big(\big(\sigma_1\omega\big)^p\sigma^p + \|F\|_{L^\infty((0,1)\times(0,\infty))}^p\Big)\psi^p(\sigma).$$

Let us choose $\omega = \omega_1^{1/p}$. Then we have

$$\int_0^1 u_\sigma^p\,dx \le M_2(\sigma)\psi^p(\sigma), \quad (4.4)$$

where

$$M_2(\sigma) = M_1\left(\sigma_1^p\sigma^p + \frac{\|F\|_{L^\infty((0,1)\times(0,\infty))}^p}{\omega_1}\right). \quad (4.5)$$

Since, for $\hat\sigma > \sigma \ge \sigma_0$,

$$\int_0^1 u_\sigma^p(t)\,dx \ge \int_{\Omega_{\hat\sigma}(t)} u_\sigma^p(t)\,dx \ge (\hat\sigma - \sigma)^p\lambda(t)^p\Omega_{\hat\sigma}(t),$$

it thus follows from (4.4) that

$$\psi(\hat\sigma) \le M_2(\sigma)(\hat\sigma - \sigma)^{-p}\psi(\sigma)^p,$$

where we have used the fact that $\lambda(t) \ge 1$.

Now we apply a Stampacchia-type L^∞ estimate, which can be found in [5].

LEMMA 4.1. *Suppose $\psi(\sigma)$ is a nonnegative, nonincreasing function on $[\sigma_0,\infty)$, and there are positive constants γ and β such that for all $\hat\sigma > \sigma \ge \sigma_0$,*

$$\psi(\hat\sigma) \le M(\sigma)(\hat\sigma - \sigma)^{-\gamma}\psi(\sigma)^{1+\beta},$$

where the function $M(\sigma)$ *is nondecreasing and satisfies*

$$0 \le \sigma^{-\gamma} M(\sigma) \le M_0$$

for all $\sigma \in [\sigma_0, \infty)$. *Then*

$$\psi(\sigma^*) = 0$$

with $\sigma^* = 2\sigma_0(1 + 2^{(1+2\beta)/\beta^2} M_0^{(1+\beta)/\beta\gamma} \psi(\sigma_0)^{(1+\beta)/\gamma})$.

Since $T > 0$ is arbitrary, it follows from Lemma 4.1, with $\gamma = p$, $\beta = p - 1$, and $M(\sigma) = M_2(\sigma)$, that there exists $\sigma \ge \sigma_0$ such that

$$u(x,t) \le \sigma\left(\sigma_1 e^{-\omega t} + 1\right) \qquad \text{a.e.} \quad x \in (0,1), \quad t \ge 0, \tag{4.6}$$

where the positive constants σ_1 and ω are defined above. Furthermore, the same arguments, using $|u_\sigma|^{p-2} u_\sigma$ with

$$u_\sigma(x,t) = \min\{0, u(x,t) + \sigma\lambda(t)\} \qquad \text{and} \qquad \lambda(t) = \sigma_2 e^{-\omega t} + 1$$

as the test function, can be applied to show that

$$u(x,t) \ge -\sigma\left(\sigma_2 e^{-\omega t} + 1\right) \qquad \text{a.e.} \quad x \in (0,1), \quad t \ge 0, \tag{4.7}$$

where

$$\sigma_2 = -\operatorname*{ess\,inf}_{x \in (0,1)} \{u_0(x)\}.$$

The above arguments are summarized, combination with Theorem 3.1, in the following theorem.

THEOREM 4.1. *Suppose* $f \in C^1(\mathbb{R})$, *the nonincreasing functions* $g_0, g_1 \in C(\mathbb{R})$ *satisfy* $g_0(0) = g_1(0) = 0$ *and the growth condition* (4.2), *and* $F \in L^\infty((0,1) \times (0,\infty))$ *and* $u_0 \in L^\infty(0,1)$. *Then the unique solution* $u(x,t)$ *of the scalar conservation law* (1.1) *satisfies*

$$-\sigma\left(\sigma_2 e^{-\omega t} + 1\right) \le u(x,t) \le \sigma\left(\sigma_1 e^{-\omega t} + 1\right)$$
$$a.e. \quad x \in (0,1), \quad \forall t \ge 0,$$

where

$$\sigma_1 = \operatorname*{ess\,sup}_{x \in (0,1)} \{u_0(x)\} \quad \textit{and} \quad \sigma_2 = -\operatorname*{ess\,inf}_{x \in (0,1)} \{u_0(x)\}$$

for some $\sigma \ge 1$ *and* $\omega > 0$ *independent of the initial condition* u_0. *In particular,*

$$\limsup_{t \to \infty} \|u(t)\|_{L^\infty} \le \sigma \overset{\text{def}}{=} \rho$$

and

$$\|u(t)\|_{L^\infty} \le \sigma\big(\|u_0\|_{L^\infty} + 1\big) \overset{\text{def}}{=} \theta \qquad \forall t \ge 0.$$

Remark 4.1. The growth condition (4.2) is satisfied by the Burgers equation, i.e., $f(u) = \frac{1}{2}u^2$, with boundary feedback laws of the form

$$g_0(u) = \beta_0|u|u \qquad \text{and} \qquad g_1(u) = \beta_1|u|u,$$

where $\beta_0, \beta_1 > 0$. In fact, for $y, \sigma \ge 0$,

$$|y + \sigma|(y + \sigma)|y|^{p-2}y \ge |y|^{p+1} + 2\sigma|y|^p$$

and

$$\int_0^y (s + \sigma)|s|^{p-2}s\,ds = \frac{1}{p+1}|y|^{p+1} + \frac{\sigma}{p}|y|^p.$$

Hence (4.2) is satisfied for sufficiently large p. The same argument applies when $y < 0$.

To prove the precompactness of the nonlinear semigroup of (1.1), we establish the following regularity estimates for the solutions with respect to time t.

LEMMA 4.2. *Assume that the conditions of Theorem* 4.1 *hold true. Then, for all* $t, \tau \ge 0$,

$$\int_t^{t+\tau} \|u(s)\|_1^2\,ds \le \Big(\|u(t)\|_{L^\infty}^2 + 2\tau \max_{s\in[t,t+\tau]} \alpha\big(\|u(s)\|_{L^\infty}\big)\Big)\Big/(2\nu), \quad (4.8)$$

where

$$\begin{aligned}\alpha(r) &= \alpha\big(r; \|F\|_{L^\infty((0,1)\times(0,\infty))}, \nu\big)\\ &= 2\nu r^2 + 2r\Big(\max_{|y|\le r}|f'(y)y|\Big) + r\|F\|_{L^\infty((0,1)\times(0,\infty))}\end{aligned} \qquad (4.9)$$

is nondecreasing in r. *Moreover, for each* $\tau > 0$, *there exists a constant* M_τ *that also depends on* $\|u_0\|_{L^\infty}$ *and* $\|F\|_{L^\infty((0,1)\times(0,\infty))}$ *such that*

$$t^2\|u_t(t)\|_{L^2}^2 + \frac{\nu}{2}\int_0^t s^2\|u_t(s)\|_1^2\,ds \le M_\tau t^2. \qquad (4.10)$$

Proof. Taking the inner product of (1.1) with $u(t)$ and by applying Theorem 4.1 and the fact that $g_i(y)y \geq 0$ for $i = 1, 2$, we have

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\|u(t)\|_{L^2} + \nu\|u(t)\|_1^2 \\
&\qquad = \int_0^1 F(t)u(t)\,dx + \nu\big(u^2(0,t) + u^2(1,t)\big) \\
&\qquad\quad - \nu\big(g_0(u(0,t))u(0,t) + g_1(u(1,t))u(1,t)\big) \\
&\qquad\quad - \int_{u(0,t)}^{u(1,t)} f'(s)s\,ds \leq \alpha\big(\|u(t)\|_{L^\infty}\big),
\end{aligned}$$

where $\alpha(r)$ is as in (4.9). Integrating this inequality, we obtain (4.8).

Next, we prove the estimate (4.10) by estimating the corresponding difference quotient. For a function $\psi(t)$, define the difference quotient

$$\psi_h(t) = \frac{\psi(t+h) - \psi(t)}{h}$$

for $h \neq 0$. Taking the difference of (1.1) at t and $t + h$ and multiplying by $u_h(t)$,

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\|u_h(t)\|_{L^2}^2 + \frac{1}{h}\int_0^1 \big(f(u(t+h)) - f(u(t))(u_h)_x(t)\big)\,dx \\
&\quad + \nu\|(u_h(t))_x\|_{L^2}^2 + \frac{1}{h}\big(f(u(1,t+h)) - f(u(1,t))\big)u_h(1,t) \\
&\quad - \big(f(u(0,t+h)) - f(u(0,t))\big)u_h(0,t) + \frac{\nu}{h}\big[\big(g_0(u(0,t+h)) \\
&\qquad - g_0(u(0,t))u_h(0,t) + \big(g_1(u(1,t+h)) - g_0(u(1,t))u_h(1,t)\big] \\
&\quad = 0.
\end{aligned}$$

Since $f \in C^1(\mathbb{R})$ and g_0, g_1 are nondecreasing, it follows from Theorem 4.1 and the Mean Value Theorem that

$$\frac{1}{2}\|u_h(t)\|_{L^2}^2 + \frac{\nu}{2}\|u_h(t)\|_1^2 \leq \left(\nu + \frac{M_2}{\nu}\right)\|u_h(t)\|_{L^2}^2$$

for some $M_2 > 0$, where we used the inequality

$$\|\phi\|_{L^\infty}^2 \leq \beta\|\phi\|_1^2, \quad \phi \in X \qquad \text{for some } \beta > 0.$$

Note that

$$\frac{d}{dt}\left(\frac{t^2}{2}\|u_h(t)\|_{L^2}^2\right) = t\|u_h(t)\|_{L^2}^2 + \frac{t^2}{2}\frac{d}{dt}\|u_h(t)\|_{L^2}^2.$$

Hence

$$\frac{t^2}{2}\|u_h(t)\|_{L^2}^2 + \frac{\nu}{2}\int_0^t s^2\|(u_h(s))_x\|_{L^2}^2\, ds \le \int_0^t (1 + M_3 s)s\|u_h(s)\|_{L^2}^2\, ds,$$

where $M_3 = \nu + M_2/\nu$. Note further that

$$\|u_h(s)\|_{L^2}^2 \le c_1\|u_h(s)\|_{X^*}\|u_h(s)\|_1.$$

Therefore, from the above estimate, we obtain

$$\frac{t^2}{2}\|u_h(t)\|_{L^2}^2 + \frac{\nu}{4}\int_0^t s^2\|u_h(s)\|_1^2\, ds \le c\int_0^t \frac{1}{\nu}(1 + M_3 s)^2\|u_h(s)\|_{X^*}^2\, ds.$$

It follows from (1.1) that

$$\begin{aligned}\langle u_t, \phi\rangle = &-\nu\big(g_0(u(0,t))\phi(0) + g_1(u(1,t))\phi(1)\big) - \nu\big(u_x(t), \phi_x\big)\\ &+ f(u(0,t))\phi(0) - f\big(u(1,t)\phi(1) + \big(f(u(t)), \phi_x\big) + (F, \phi)\end{aligned}$$

for $\phi \in X$. It thus follows from Theorem 4.1 that there exists a constant $M_4 > 0$ such that

$$\int_0^t s^2\|u_t(s)\|_{X^*}^2\, ds \le \int_0^t s^2\big(M_4\|u(s)\|_1^2 + \|F(s)\|_{L^\infty}^2\big)\, ds.$$

Thus, the estimate (4.10) follows by using the relations between difference quotient and derivative (see, e.g., [8]).

The last estimate follows from

$$\frac{t^2}{2}\|u(t)\|_1^2 = \int_0^t s\|u(s)\|_1^2\, ds + \int_0^t s^2(u_t(s), u(s))_1\, ds.$$

Therefore the proof of the lemma is completed.

Now we can prove that the trajectories are precompact in L^∞.

THEOREM 4.2. *Assume that the conditions in Theorem* 4.1 *hold true. Then there exists a constant* ρ_1 *independent of the initial state* u_0 *such that for each* $\varepsilon > 0$, *there exists a* $t_0 = t_0(\|u_0\|_{L^\infty})$ *such that*

$$\|u(t)\|_1 \le \rho_1 + \varepsilon \qquad \forall t \ge t_0\big(\|u_0\|_{L^\infty}\big). \tag{4.11}$$

Moreover,

$$\|u(t)\|_1 = O(t^{-1/2}) \qquad \text{as } t \to 0^+. \tag{4.12}$$

Proof. Multiplying (1.1) by $-u_{xx}$ and integrating with respect to x over $(0,1)$,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_x(t)\|_{L^2}^2 + \int_0^{u(0,t)} g_0(y)\,dy + \int_0^{u(1,t)} g_1(y)\,dy\right) + \nu\|u_{xx}(t)\|_{L^2}^2 \\
&\quad = \int_0^1 f'(u(t))u_x(t)u_{xx}(t)\,dx - \int_0^1 F(t)u_{xx}(t)\,dx \\
&\quad \le \left(\max_{|y|\le\|u(t)\|_{L^\infty}} |f'(y)|\right)\|u_x(t)\|_{L^2}\|u_{xx}\|_{L^2} + \|F\|_{L^\infty}\|u_{xx}\|_{L^2},
\end{aligned}$$

where, by Lemma 4.2, $d/dt \int_0^{u(i,t)} g_i(y)\,dy = g_i(u(i,t)u_t(i,t)$ exists a.e. $t > 0$. By using Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_x(t)\|_{L^2}^2 + \int_0^{u(0,t)} g_0(y)\,dy + \int_0^{u(1,t)} g_1(y)\,dy\right) \\
&\quad \le \frac{d}{dt}\left(\frac{1}{2}\|u_x(t)\|_{L^2}^2 + \int_0^{u(0,t)} g_0(y)\,dy + \int_0^{u(1,t)} g_1(y)\,dy\right) \\
&\qquad + \frac{\nu}{2}\|u_{xx}(t)\|_{L^2}^2 \\
&\quad \le \left(\max_{|y|\le\|u(t)\|_{L^\infty}} |f'(y)|^2\right)\|u_x(t)\|_{L^2}^2 \Big/ \nu + \|F(t)\|_{L^\infty}^2/\nu.
\end{aligned}$$

We have

$$x(t) = \frac{1}{2}\|u_x(t)\|_{L^2}^2 + \int_0^{u(0,t)} g_0(y)\,dy + \int_0^{u(1,t)} g_1(y)\,dy$$

and

$$q(t) = \left(\max_{|y|\le\|u(t)\|_{L^\infty}} |f'(y)|^2\right)\|u_x(t)\|_{L^2}^2 \Big/ \nu + \|F(t)\|_{L^\infty}^2/\nu;$$

then

$$\frac{dz(t)}{dt} \le q(t) \qquad \forall t \ge 0. \tag{4.13}$$

But

$$\int_t^{t+\tau} z(s)\,ds \le \frac{1}{2}\int_t^{t+\tau} \|u(s)\|_1^2\,ds + \tau \max_{s\in[t,t+\tau]}\left[\|u(s)\|_{L^\infty}\left(\max_{|y|\le\|u(s)\|_{L^\infty}}|g_0(y)| + \max_{|y|\le\|u(s)\|_{L^\infty}}|g_1(y)|\right)\right], \tag{4.14}$$

and

$$\int_t^{t+\tau} q(s)\,ds \le \max_{s\in[t,t+\tau]}\left(\max_{|y|\le\|u(s)\|_{L^\infty}}|f'(y)|^2\right)\left(\int_t^{t+\tau}\|u(s)\|_1^2\,ds\right)\Big/\nu + \tau\|F\|_{L^\infty((0,1)\times(0,\infty))}/\nu \tag{4.15}$$

It follows from Theorem 4.1 that for each $\delta > 0$, there exists $t_0 = t_0(\|u_0\|_{L^\infty})$ such that

$$\|u(t)\|_{L^\infty} \le \rho + \delta \qquad \forall t \ge t_0,$$

where ρ is as in Theorem 4.1. Then, by Lemma 4.2,

$$\int_t^{t+\tau}\|u(s)\|_1^2\,ds \le b(\rho+\delta;\tau) \qquad \forall t \ge t_0,\ \tau \ge 0,$$

where

$$b(r;\tau) = \left(r^2 + 2\tau\alpha(r)\right)/(2\nu),$$

and then

$$\int_t^{t+\tau} z(s)\,ds \le a_1(\rho+\delta;\tau) \quad \text{and} \quad \int_t^{t+\tau} q(s)\,ds \le a_2(\rho+\delta;\tau) \qquad \forall t \ge t_0, \quad \tau \ge 0, \tag{4.16}$$

where

$$a_1(r;\tau) = \left(r^2 + 2\tau\alpha(r)\right)/(4\nu) + \tau r\left(\max_{|y|\le r}|g_0(y)| + \max_{|y|\le r}|g_1(y)|\right), \tag{4.17}$$

and

$$a_2(r;\tau) = \left(\max_{|y|\le r}|f'(y)|^2\right)\left(r^2 + 2\tau\alpha(r)\right)\Big/(2\nu^2) + \tau\|F\|_{L^\infty((0,1)\times(0,\infty))}/\nu. \tag{4.18}$$

By Uniform Gronwall's Lemma, we obtain

$$z(t) \le \frac{a_1(\rho+\delta;\tau)}{\tau} + a_2(\rho+\delta;\tau) \qquad \forall\tau_*,\ t\ge t_0+\tau.$$

Hence

$$\|u(t)\|_1^2 \le 2\|u(t)\|_{L^\infty}^2 + \|u_x(t)\|_{L^2}^2 \le \tilde{\alpha}(\rho+\delta;\tau) \qquad \forall\tau\ge 0,\ t\ge t_0+\tau, \tag{4.19}$$

where

$$\tilde{\alpha}(r;\tau) = 2r^2 + 2r\left(\max_{|y|\le r}|g_0(y)| + \max_{|y|\le r}|g_1(y)|\right) + \frac{2a_1(r;\tau)}{\tau} + 2a_2(r;\tau).$$

Therefore (4.11) follows from (4.19) with $\rho_1 = (\tilde{\alpha}(\rho;1))^{1/2}$.

By the same token, we can prove

$$\|u(t)\|_1^2 \le \tilde{\alpha}(\theta;\tau) = O(\tau^{-1/2}) \qquad \forall\tau\ge 0,\ t\ge\tau,$$

where θ is as in Theorem 4.1. Hence (4.12) is proved. Therefore the proof of Theorem 4.2 is completed.

The above arguments are summarized as the following main theorem.

THEOREM 4.3. *Under the assumptions of Theorem* 4.1, *the scalar conservation law* (1.1) *has a nonempty compact global attractor* $\mathscr{A}$ *in* $L^\infty(0,1)$. *Moreover*, $\mathscr{A}$ *is also connected and is contained in a bounded set of* $H^1(0,1)$.

ACKNOWLEDGMENTS

The authors are grateful to Prof. John A. Burns for his encouragement and helpful discussions. In fact, Prof. Burns established a connection between two authors who were working on similar problems separately in different directions, leading to the present form of this paper.

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